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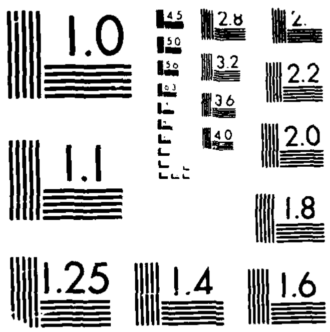

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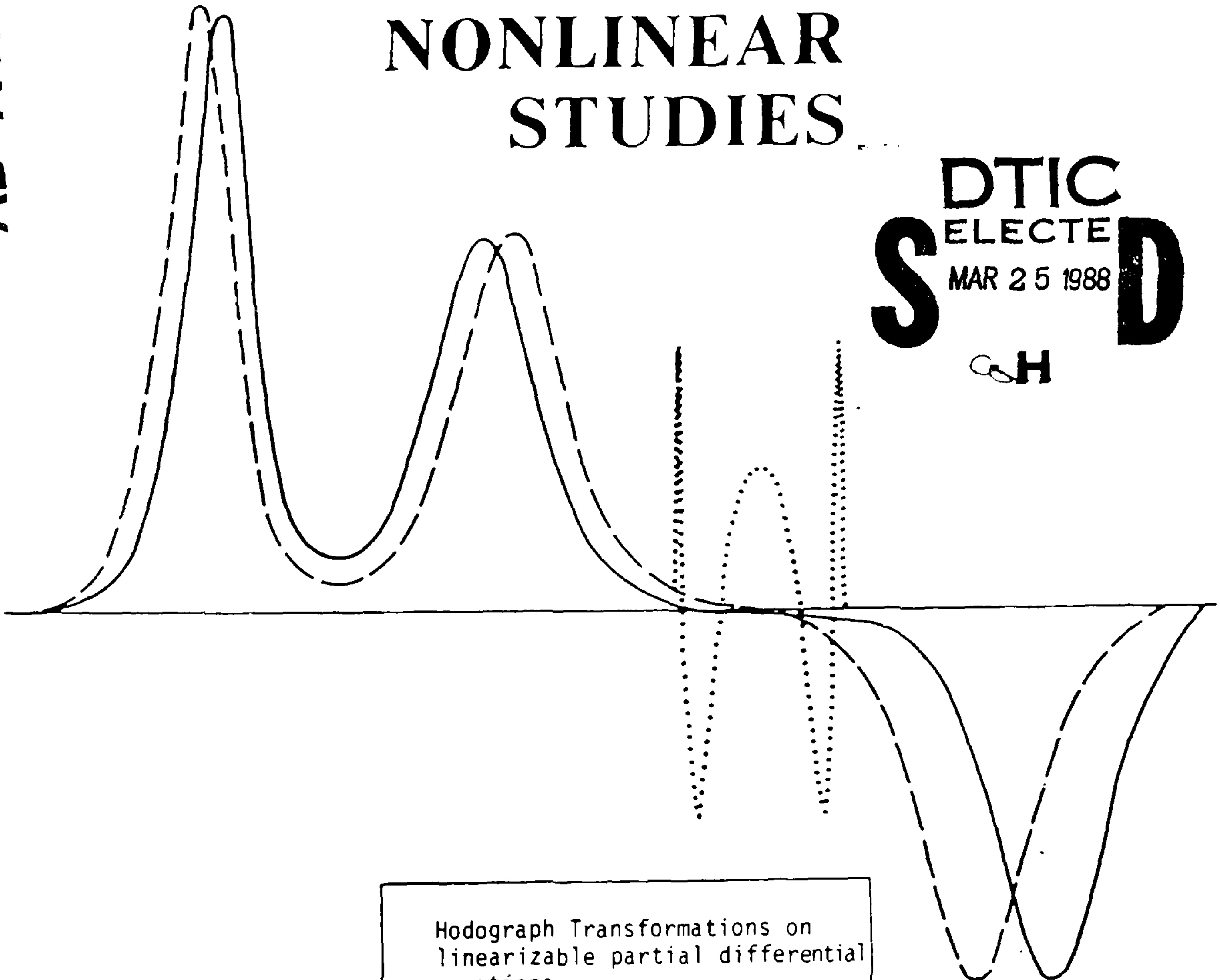

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HODOGRAPH TRANSFORMATIONS ON LINEARIZABLE PARTIAL DIFFERENTIAL EQUATIONS

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P.A. Clarkson[1], A.S. Fokas and M.J. Ablowitz

Department of Mathematics and Computer Science

and

Institute for Nonlinear Studies

Clarkson University

Potsdam, New York 13676 U.S.A.

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[1]Present address: Department of Mathematics, Birmingham University
Birmingham B15 2TT, England

ABSTRACT

In this paper ~~we~~ develop an algorithmic method for transforming quasilinear partial differential equations of ~~the~~ a certain form $u_t = g(u)u_{nx} + f(u,u_x,\ldots,u_{(n-1)x})$, $u_{mx} \equiv \partial^m u/\partial x^m$, where $dg/du \not\equiv 0$, into semilinear equations (i.e., equations of the above form with $g(u) = 1$). This crucially involves the use of hodograph transformations (i.e., transformations which involve the interchange of dependent and independent variables). Furthermore, we find the most general quasilinear equation of the above form which can be mapped via a hodograph transformation to a semilinear form.

This algorithm provides a method for establishing whether a given quasilinear equation is linearizable; i.e., is solvable in terms of either a linear partial differential equation or of a linear integral equation. In particular, we use this method to show how the Painlevé tests may be applied to quasilinear equations. This appears to resolve the problem that solutions of linearizable quasilinear partial differential equations, such as the Harry-Dym equation $u_t = (u^{-1/2})_{xxx}$, typically have movable fractional powers and so do not directly pass the Painlevé tests.

I. INTRODUCTION

Recently there has been considerable interest in the solution of certain physically significant, nonlinear partial differential equations. It turns out that the solutions of these equations may be expressed in terms of the solution of linear equations (either linear integral equations or linear partial differential equations). In 1967, Gardner, Greene, Kruskal and Miura [1] associated the solution of the Korteweg-de Vries (KdV) equation with the time independent Schrödinger equation and showed, using ideas from the theory of direct and inverse scattering, that the Cauchy problem for the KdV equation (for initial data on the line which decays sufficiently rapidly), could be solved in terms of the solution of a linear integral equation. Subsequently, this novelty was developed into a new method of mathematical physics, often referred to as the inverse scattering transform (I.S.T.), which has led to the solution of numerous evolution equations (see, for example, [2] for details). These nonlinear evolution equations have arisen in many branches of physics including water waves, stratified fluids, plasma physics, statistical mechanics and quantum field theory. Previous to the KdV equation, the first physically interesting nonlinear partial differential equation which was solved in terms of a linear partial differential equation was Burgers' equation

$$u_t = u_{xx} + 2uu_x, \tag{1.1}$$

which was mapped into the linear heat equation via the Cole-Hopf transformation [3].

Partial differential equations which can either be solved by an appropriate I.S.T. scheme or by a transformation to a linear partial differential equation are said to be linearizable. The most well known linearizable partial differential equations are of the form

$$u_t = u_{nx} + f(u, u_x, \ldots, u_{(n-1)x}),\ n \geq 2,\ u_{nx} \equiv \frac{\partial^n u}{\partial x^n} \tag{1.2}$$

<u>Definition 1.1</u> A partial differential equation is said to be <u>semilinear</u> if it is of the form (1.2).

There also exist linearizable equations of the form

$$u_t = g(u)u_{nx} + f(u, u_x, \ldots, u_{(n-1)x}),\ n \geq 2, \tag{1.3}$$

where $dg/du \not\equiv 0$.

<u>Definition 1.2</u> A partial differential equation is said to be <u>quasilinear</u> if it is of the form (1.3).

Well known examples of quasilinear linearizable equations include an equation studied in [4],

$$u_t = (u^{-2}u_x)_x + \alpha u^{-2}u_x, \tag{1.4}$$

where α is an arbitrary constant and the Harry-Dym equation (Kruskal [5])

$$u_t = 2(u^{-1/2})_{xxx}, \tag{1.5}$$

which is known to be linearizable [6] (see also [2b]).

Fokas and Yortsos [4] considered second order quasilinear partial differential equations using the symmetry approach of Fokas [7]. They showed that the most general equation of the form

$$u_t = g(u)u_{xx} + f(u,u_x), \tag{1.6}$$

which is linearizable is the equivalent to the equation (1.4), which via an extended hodograph transformation is mapped to the Burgers' equation. Similarly, it is known that the Harry-Dym equation (1.5) can be transformed either into the KdV equation (see, for example, [2b] or [8]), or the MKdV equation (see, for example, Kawamoto [9]). The notions of equivalence and hodograph transformations are defined below:

Definition 1.3 Two partial differential equations are equivalent if one can be obtained from the other by a transformation involving the dependent variables $u = \phi(v)$ and/or the introduction of a potential variable ($u = v_x$ or $u_x = v$).

For example, the Burgers' equation is equivalent to the heat equation.

Definition 1.4 A pure hodograph transformation is a transformation of the form

$$\tau = t, \quad \xi = u(x,t). \tag{1.7}$$

Definition 1.5 An extended hodograph transformation is a transformation of the form

$$\tau = t, \quad \xi = \int^{x} (u(x',t))dx'. \tag{1.8}$$

The above discussion naturally motivates the following questions: Equation (1.4) is a quasilinear analogue, via an extended hodograph transformation, of Burgers' equation. Similarly, the Harry-Dym equation (1.5) is a quasilinear analogue of the MKdV equation.

i) Is there an algorithmic method of finding a quasilinear analogue of any semilinear equation?

ii) Is the associated quasilinear equation unique?

iii) Conversely, given a quasilinear equation, is there an algorithmic method of finding whether it can be mapped to a semilinear equation as well as finding this semilinear equation?

In this paper we consider the above questions for semilinear and quasilinear equations (1.2) and (1.3) respectively. The answer to question i) is affirmative. Also, the associated quasilinear equation is unique, since extended and pure hodograph transformations yield equivalent quasilinear equations. Furthermore, we find the most general equation of the form (1.3) which can be mapped via an extended hodograph transformation to a semilinear form.

The above results are of some interest in establishing whether an equation is a candidate for linearization. Suppose that one is interested in investigating whether a given quasilinear equation is linearizable. We propose the following algorithmic procedure (see §III);

1. Put the equation into its potential canonical form

$$v_t = v_x^{-n} v_{nx} + H(v_x, v_{xx}, \ldots, v_{(n-1)x}), \tag{1.9}$$

by using the transformation $v_x = g^{-1/n}(u)$.

2. Apply a pure hodograph transformation to equation (1.9). If equation (1.9) is transformable to a semilinear equation, it will become

$$\eta_t = \eta_{n\xi} + \tilde{H}(\eta_\xi, \eta_{\xi\xi}, \ldots, \eta_{(n-1)x}). \tag{1.10}$$

3. Investigate whether equation (1.10) is linearizable. This is easier than investigating whether (1.2) is linearizable directly. The reason for this is twofold. First, for at least third order equations there is a complete classification of all linearizable equations. Within equivalence, there exist only six such equations (see below). Hence one needs to study if there exists an equivalence transformation to map equation (1.10) with n = 3, to one of these six canonical equations. Second, for equations with $n \geq 4$ one may investigate the question of linearization via the Painlevé test. The Painlevé approach is reviewed below. Here we only point out that quasilinear partial differential equations do not appear suitable for applying the Painlevé test. Ramani,

Dorizzi and Grammaticos [10] (see also [11] and the references therein) introduced the notion of "weak-Painlevé" in order to deal with equations such as the Harry-Dym equation which are linearizable after a change of variables. However, the higher KdV equation $u_t = u_{xxx} + u^3u_x$, although not thought to be linearizable (since it has only three independent polynomial conservation laws of a certain type [12]), is also "weak-Painlevé" [13]. Therefore the "weak-Painlevé" concept does not distinguish between a linearizable and a non linearizable equation.

We point out that one often finds in the literature claims of "new" third order linearizable equations. These equations, using the notion of equivalence can be mapped via a pure hodograph transformation to one of the six canonical equations mentioned above.

The above algorithmic approach is useful provided that a given linearizable quasilinear equation can be mapped to a semilinear form. The above approach will fail if there exist linearizable quasilinear equations which can not be mapped to a semilinear from. It is shown in [4] that such equations do not exist for at least n = 2. The question of whether such equations exist for $n \geq 3$ remains open. Important results in this direction can be found in [43].

IA. Classification of third order equations

Svinolupov, Sokolov and Yamilov [14] have claimed that the only third order semilinear partial differential equations which are linearizable are equivalent to the following six equations:

$$u_t = u_{xxx} + \gamma u_x, \tag{1.11}$$

$$u_t = u_{xxx} + uu_x + \gamma u_x, \tag{1.12}$$

$$u_t = u_{xxx} + u^2 u_x + \gamma u_x, \tag{1.13}$$

$$u_t = u_{xxx} - \frac{1}{8}u_x^3 + (\alpha e^u + \beta e^{-u})u_x + \gamma u_x, \tag{1.14}$$

$$u_t = u_{xxx} - \frac{3}{2}u_x u_{xx}^2 (1 + u_x^2)^{-1} - \frac{3}{2}P(u)(u_x^2 + 1)u_x + \gamma u_x, \tag{1.15}$$

$$u_t = u_{xxx} - \frac{3}{2}u_{xx}^2 u_x^{-1} + \alpha u_x^{-1} - \frac{3}{2}P(u)u_x^2 + \gamma u_x, \tag{1.16}$$

where

$$\left(\frac{dP}{du}\right)^2 = 4P^3 - \delta P - \epsilon, \tag{1.17}$$

and α, β, γ, δ and ϵ are arbitrary constants. Equation (1.11) is a linear partial differential equation which is sometimes referred to as the Airy equation in moving coordinates; equation (1.12) is the KdV equation, which was the first equation to be solved by I.S.T.[1]; equation (1.13) is the Modified KdV (MKdV) equation, also solvable by I.S.T. [15]; equation (1.14) is the Calogero-Degasperis-Fokas (CDF) equation [7],[16],equations (1.15) and (1.16) are as yet unnamed and involve the Weierstrass elliptic function $P(u)$. We note that the CDF equation can be put into rational form: let $v = e^{u/2}$.

$$v_t = v_{xxx} - \frac{3}{2}(v_x^2/v)_x + (\alpha v^2 + \beta v^{-2} + \gamma)v_x . \tag{1.18}$$

Alternatively, provided that $\alpha = \beta = -2\gamma$ (if $\alpha\beta \neq 0$, then one can rescale and translate the variables in (1.14) so that this holds), let $q = \sinh(u/2)$ to obtain

$$q_t = q_{xxx} - \frac{3}{2}[qq_x^2/(1 + q^2)]_x + 4\alpha q^2 q_x. \tag{1.19}$$

(Equation (1.19) is sometimes referred to as the 'deformed MKdV' equation [17] or the modified MKdV [18], though it is equivalent to the CDF equation.) We also note that both equations (1.15) and (1.16) can be put into rational form by the substitution $v = P(u)$.

IB. <u>The Painlevé Tests</u>

The Painlevé ODE test, as formulated by Ablowitz, Ramani and Segur [19] and Hastings and McLeod [20] asserts that every ordinary differential equation which arises as a similarity reduction of a partial differential equation solvable by inverse scattering is of Painlevé type; that is, it has no movable singularities except poles, perhaps after a transformation of variables. Ablowitz, Ramani and Segur [19b] and McLeod and Olver [21] have given proofs of the Painlevé ODE test under certain restrictions. Subsequently, Weiss, Tabor and Carnevale [22] developed the Painlevé PDE test as a method of applying the Painlevé ODE test directly to a given partial differential equation, without having to study any similarity reductions (which may not exist anyway). A partial differential equation is said to possess the Painlevé property if its solutions are "single-valued" in the neighborhood of noncharacteristic movable singularity manifolds. These Painlevé tests have proved to be a useful criterion for the identification of linearizable partial differential equations. The method introduced by Weiss, Tabor and Carnevale (with simplifications due to Kruskal [23]), involves seeking solutions of a given partial differential equation in the form

$$u(x,t) = \phi^{p}(x,t)\sum_{j=0}^{\infty} u_j(t)\phi^{j}(x,t), \tag{1.20a}$$

with

$$\phi(x,t) = x + f(t), \tag{1.20b}$$

where f(t) is an arbitrary, analytic function of t and $u_j(t)$, $j = 0,1,2,\ldots$, are analytic functions of t, in the neighborhood of a noncharacteristic movable singularity manifold defined by $\phi = 0$. Essentially, if a given partial differential equation possesses solutions of the form (1.20) where p is an integer and with the requisite number of arbitrary functions as required by the Cauchy-Kowalevski theorem, then the partial differential equation is said to pass the Painlevé PDE test.

However, the application of the Painlevé tests to quasilinear partial differential equations is not as straightforward. For example, consider the Harry-Dym equation (Kruskal [5])

$$u_t = 2(u^{-1/2})_{xxx}, \tag{1.21}$$

which is known to be linearizable [6] (see also [2b]). Then (1.21) does not <u>directly</u> (i.e., without a transformation of variables) pass the Painlevé PDE test since it has an expansion of the form

$$u(x,t) = \phi^{-4/3}(x,t) \sum_{j=0}^{\infty} u_j(t)\phi^{1/3}(x,t), \tag{1.22}$$

with $\phi(x,t) = x + f(t)$, in the neighborhood of a noncharacteristic movable singularity manifold defined by $\phi = 0$ and so has movable cube roots (see Weiss [24] for details). If an equation has an expansion of the form

$$u(x,t) = \phi^{p/r}(x,t) \sum_{j=0}^{\infty} u_j(t)\phi^{j/r}(x,t), \tag{1.23}$$

where p and r are integers determined from the leading order analysis, then the equation is said to be "<u>weak-Painlevé</u>". However, as was pointed out earlier, the non linearizable equation $u_t = u_{xxx} + u^3 u_x$ is also weak-Painlevé.

II. <u>SECOND AND THIRD ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS</u>

An extended hodograph transformation comprises of the change of variables $u \to v_x = \phi(u(x,t))$ followed by a pure hodograph transformation, and therefore these transformations are simply related.

We first consider the pure hodograph transformation in more detail.

Let

$$t = \tau, \quad x = \eta(\xi,\tau) \tag{2.1}$$

Then using (1.7),

$$\partial_x = \xi_x \partial_\xi + \tau_x \partial_\tau = u_x \partial_\xi, \tag{2.2a}$$

$$\partial_t = \xi_t \partial_\xi + \tau_t \partial_\tau = u_t \partial_\xi + \partial_\tau. \tag{2.2b}$$

Therefore the Jacobian of this transformation is u_x. Similarly for the inverse transformation (2.1) we have

$$\partial_\xi = x_\xi \partial_x + t_\xi \partial_t = \eta_\xi \partial_x, \tag{2.3a}$$

$$\partial_\tau = x_\tau \partial_x + t_\tau \partial_t = \eta_\tau \partial_x + \partial_t. \tag{2.3b}$$

Under a pure hodograph transformation, derivatives transform as follows

$$u_x = \eta_\xi^{-1}, \quad u_t = -\eta_\tau \eta_\xi^{-1}, \tag{2.4a}$$

$$u_{xx} = -\eta_{\xi\xi}\eta_\xi^{-3}, \quad u_{xxx} = -\eta_{\xi\xi\xi}\eta_\xi^{-4} + 3\eta_{\xi\xi}^2\eta_\xi^{-5}, \tag{2.4b}$$

or inversely

$$\eta_\xi = u_x^{-1}, \quad \eta_\tau = -u_t u_x^{-1} \tag{2.5a}$$

$$\eta_{\xi\xi} = -u_{xx}u_x^{-3}, \quad \eta_{\xi\xi\xi} = -u_{xxx}u_x^{-4} + 3u_{xx}^{-2}u_x^{-5}, \tag{2.5b}$$

Therefore the linear partial differential equation

$$u_t = u_{xxx}, \tag{2.6}$$

under a pure hodograph transforms to

$$\eta_\tau = \eta_{\xi\xi\xi}\eta_\xi^{-3} - 3\eta_{\xi\xi}^2\eta_\xi^{-4}. \tag{2.7}$$

Note that if one applies a pure hodograph transformation to a partial differential equation in potential form (that is an equation which does <u>not</u> depend explicitly on the dependent variable) which also does not depend explicitly on the independent variables, then the resulting equation is also in potential form with no explicit dependence on the independent variables. Therefore, before applying a pure hodograph transformation to a given partial differential equation, we shall put the equation into canonical potential form.

We now consider second order quasilinear partial differential equations.

IIA. SECOND ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

The most general second order, quasilinear partial differential equation of the form

$$u_t = g(u)u_{xx} + f(u,u_x), \tag{2.8}$$

with $dg/du \neq 0$, which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form

$$S_\tau = S_{\xi\xi} + G(S, S_\xi), \tag{2.9}$$

is given by

$$u_t = g(u)u_{xx} + \left(\frac{gg''}{g'} - \frac{g'}{2}\right)u_x^2 + b'(u)u_x, \tag{2.10}$$

where $' \equiv d/du$, and $g(u)$ and $b(u)$ are arbitrary functions which are twice and once differentiable, respectively. Furthermore, equation (2.9) is equivalent to the equation

$$v_t = v_x^{-2}v_{xx} + H(v_x), \tag{2.11}$$

which is transformed via a pure hodograph transformation to

$$\eta_\tau = \eta_{\xi\xi} - \eta_\xi H(\eta_\xi^{-1}) . \tag{2.12}$$

<u>Proof</u>

In equation (2.8) we make the transformation

$$\tau = t, \quad \xi = F(x,t), \quad \eta(\xi,\tau) = u(x,t),$$

then (2.8) becomes

$$[illegible] = g(u)F_x^2 [illegible] + (gF_{xx} - F_t) [illegible] + f([illegible], [illegible] F_x).$$

Now choose F such that

$$gF_x^2 = 1, \quad \text{i.e., } F_x = g^{-1/2}, \tag{2.13a}$$

$$F_t = A(u,u_x), \tag{2.13b}$$

where $A(u,u_x)$ is such that the compatibility of (2.13) (i.e., $F_{xt} = F_{tx}$) implies (2.8). Therefore

$$-\frac{1}{3}g^{-3/2}g'u_t = A_u u_x + A_{u_x} u_{xx}, \tag{2.14}$$

where $A_u = \partial A/\partial u$, $A_{u_x} = \partial A/\partial u_x$; using (2.8)

$$-\frac{1}{3}g^{-1/2}g'u_{xx} + gf(u,u_x) = A_u u_x + A_{u_x} u_{xx}, \tag{2.15}$$

Equating coefficients of u_{xx} to zero in (2.15), it is seen that

$$A(u,u_x) = -\frac{1}{3}g^{-1/2}g'u_x + a(u), \tag{2.16}$$

where $a(u)$ is an arbitrary function. Also from (2.15)

$$A_u u_x = -\frac{1}{3}g^{-3/2}g'f(u,u_x). \tag{2.17}$$

Therefore, from equations (2.16) and (2.17) we find that

$$f(u,u_x) = (\frac{gg''}{g'} - \frac{g'}{2})u_x^2 + b'(u)u_x, \tag{2.18}$$

where b(u) is an arbitrary function. Hence, it follows that the most general equation of the form (2.17) which is transformed via the extended hodograph transformation

$$\tau = t, \quad \xi = \int^x g^{-1/2}(u(x',t))dx'$$

into a semilinear partial differential equation has the form

$$u_t = g(u)u_{xx} + (\frac{gg''}{g'} - \frac{g'}{2})u_x^2 + b'(u)u_x. \tag{2.19}$$

We now wish to transform (2.19) into semilinear form. Our algorithm is to put (2.19) into a canonical (potential form) partial differential equation and then apply a pure hodograph transformation to convert the canonical equation into a semilinear equation. In (2.19) we make the transformation $g(u) = v_x^{-2}$ and obtain

$$v_t = v_x^{-2}v_{xx} + H(v_x), \tag{2.20}$$

where H is expressible in terms of b. Equation (2.20) is the canonical equation (since all equations of the form (2.19) are equivalent to (2.20)). It is essential that the ratio of the coefficients of v_{xx} and v_t in (2.20) is v_x^{-2} in order that the quasilinear equation is transformed into a semilinear one via a pure hodograph transformation.

Finally, applying a pure hodograph transformation to (2.20), we obtain

$$\eta_\tau = \eta_{\xi\xi} - \eta_\xi H(\eta^{-1}), \tag{2.21}$$

as required.

Therefore in summary, in order to determine which equations of the form

$$u_t = g(u)u_{xx} + f(u,u_x), \tag{2.22}$$

where $\frac{dg}{du} \neq 0$ and $f(u,u_x)$ is a rational function of u and u_x, are linearizable, it is sufficient to consider the canonical equation

$$v_t = v_x^{-2} v_{xx} + H(v_x), \tag{2.23}$$

where $H(v_x)$ is a rational function of v_x. Applying a pure hodograph transformation to (2.23) yields

$$\eta_\tau = \eta_{\xi\xi} - \eta_\xi H(\eta_\xi^{-1}) .$$

This can be put into non-potential form by making the transformation $w = \eta_\xi$, hence

$$w_\tau = w_{\xi\xi} + h(w)w_\xi , \tag{2.24}$$

where

$$h(w) = -\frac{d}{dw}[wH(1/w)]. \tag{2.25}$$

It is shown in Appendix A that equation (2.24) can pass the Painlevé tests if and only if

$$h(w) = 2\alpha w + \beta,$$

where α and β are constants. Hence from (2.25),

$$H(w) = \alpha w^{-1} + \beta. \tag{2.26}$$

Therefore, this suggests that the most general partial differential equation of the form (2.22) which is linearizable is equivalent to the equation

$$u_t = (u^{-2}u_x)_x + \alpha u^{-2}u_x. \tag{2.27}$$

We use the word "suggests" because we are aware that the Painlevé tests have not yet been proven, though there is considerable evidence suggesting their validity. This completes the "proof" of the result first obtained by Fokas and Yortsos [4]. However, the method in the present paper is somewhat simpler than that used in [4] and is easily generalizable to higher order quasilinear partial differential equations.

IIB. THIRD ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

Proposition 2.2

The most general third order, quasilinear partial differential equation of the form

$$u_t = g(u)u_{xxx} + f(u,u_x,u_{xx}), \quad \frac{dg}{du} \neq 0, \tag{2.28}$$

which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form

$$S_\tau = S_{\xi\xi\xi} + G(S,S_\xi,S_{\xi\xi}), \tag{2.29}$$

is given by

$$u_t = g(u)u_{xxx} + B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx}$$

$$+ \left(\frac{g''}{g'} - \frac{4g'}{3g}\right)B(u,u_x)u_x + \left(\frac{gg''}{g'} - \frac{g'}{3}\right)u_x u_{xx}, \tag{2.30}$$

where $B_u := \partial B/\partial u$, $B_{u_x} := \partial B/\partial u_x$, prime denotes derivative with respect to u, and $g(u)$ and $B(u,u_x)$ are arbitrary functions. Furthermore, equation (2.29) is equivalent to the equation

$$v_t = v_x^{-3}v_{xxx} + H(v_x,v_{xx}), \tag{2.31}$$

which is transformed via a pure hodograph transformation to

$$\eta_\tau = \eta_{\xi\xi\xi} - \eta_\xi H(\eta_\xi^{-1}, -\eta_{\xi\xi}\eta_\xi^{-3}). \tag{2.32}$$

<u>Proof.</u>

In equation (2.28) we make the transformation

$$\tau = t, \ \xi = F(x,t), \ \eta(\xi,\tau) = u(x,t),$$

then (2.28) becomes

$$\eta_\tau = g(u)F_x^3\eta_{\xi\xi\xi} + 3gF_xF_{xx}\eta_{\xi\xi} + (gF_{xxx} - F_t)\eta_\xi$$

$$+ f(\eta, \eta_\xi F_x, F_x^2\eta_{\xi\xi} + F_{xx}\eta_\xi^2).$$

Now choose F such that

$$gF_x^3 = 1, \text{ i.e., } F_x = g^{-1/3}, \tag{2.33a}$$

$$F_t = A(u,u_x,u_{xx}), \tag{2.33b}$$

where $A(u,u_x,u_{xx})$ is such that the compatibility of (2.33) (i.e., $F_{xt} = F_{tx}$) implies (2.28). Therefore

$$-\frac{1}{3}g^{-4/3}g'u_t = A_u u_x + A_{u_x}u_{xx} + A_{u_{xx}}u_{xxx},$$

or using (2.28)

$$-\frac{1}{3}g^{-1/3}g'u_{xxx} - \frac{1}{3}g^{-4/3}g'f(u,u_x,u_{xx})$$

$$= A_u u_x + A_{u_x}u_{xx} + A_{u_{xx}}u_{xxx} \qquad (2.34)$$

By collecting terms and equating the coefficient of u_{xxx} to zero in (2.34), it is seen that

$$A(u,u_x,u_{xx}) = -\frac{1}{3}g^{-1/3}g'u_{xx} + a(u,u_x), \qquad (2.35)$$

where $a(u,u_x)$ is an arbitrary function. Also

$$A_u u_x + A_{u_x}u_{xx} = -\frac{1}{3}g^{-4/3}g'u_x f(u,u_x,u_{xx}). \qquad (2.36)$$

Therefore, from equations (2.35) and (2.36) we find that

$$f(u,u_x,u_{xx}) = -3(g^{4/3}/g')[a_u u_x + a_u u_{xx}] + (\frac{gg''}{g'} - \frac{g'}{3})u_x^2 u_{xx},$$

$$= B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx} + (\frac{g''}{g'} - \frac{4g'}{3g})B(u,u_x)u_x$$

$$+ (\frac{gg''}{g'} - \frac{g'}{3})u_x^2 u_{xx}, \qquad (2.37)$$

where $B(u,u_x) := -3(g^{4/3}/g')a(u,u_x)$. Hence, it follows that the most general equation of the form (2.36) which is transformed via the extended hodograph transformation

$$\tau = t, \ \xi = \int^x g^{-1/3}(u(x',t))dx'$$

into a semilinear partial differential equation has the form

$$u_t = g(u)u_{xxx} + B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx}$$

$$+ \left(\frac{g''}{g'} - \frac{4g'}{3g}\right)B(u,u_x)u_x + \left(\frac{gg''}{g'} - \frac{g'}{3}\right)u_x u_{xx}, \tag{2.38}$$

In (2.38), make the transformation $g(u) = v_x^{-3}$, then we obtain

$$v_t = v_x^{-3}v_{xxx} + H(v_x,v_{xx}), \tag{2.39}$$

where $H(v_x,v_{xx})$ is expressible in terms of $B(u,u_x)$ and $g(u)$. Therefore, (2.39) is the canonical equation (again, since all equations of the form (2.38) are equivalent to (2.39)). Finally, applying a pure hodograph transformation to (2.39), we obtain

$$r_t = r_{\xi\xi\xi} - r_\xi H(r_\xi^{-1}, -r_{\xi\xi}r_\xi^{-3}), \tag{2.40}$$

as required.

Thus proposition 2.2 provides an algorithmic method of transforming the quasilinear partial differential equation

$$u_t = g(u)u_{xxx} + f(u,u_x,u_{xx}) \tag{2.41a}$$

where

$$f(u,u_x,u_{xx}) = g(u)u_{xxx} + B_u(u,u_x)u_x + B_{u_x}(u,u_x)u_{xx}$$

$$+ \left(\frac{g''}{g'} - \frac{4g'}{3g}\right)B(u,u_x)u_x + \left(\frac{gg''}{g'} - \frac{g'}{3}\right)u_x u_{xx}, \qquad (2.41b)$$

into a semilinear partial differential equation; i.e.

1. Put equation (2.41) into the potential canonical form by making the transformation $v_x = g^{-1/3}(u)$; hence we obtain

$$v_t = v_x^{-3}v_{xxx} + H(v_x, v_{xx}) \qquad (2.42)$$

2. Apply a pure hodograph transformation to equation (2.42); hence we obtain

$$\eta_\tau = \eta_{\xi\xi\xi} - \eta_\xi H(\eta_\xi^{-1}, -\eta_{\xi\xi}\eta_\xi^{-3}). \qquad (2.43)$$

3. The resulting partial differential equation will be in potential form and usually one first puts the equation into nonpotential form by making the transformation $w = \eta_\xi$. Furthermore, if the resulting semilinear partial differential equation is linearizable, then it can be expected to be equivalent to one of the six partial differential equations given by Svinolupov, Sokolov and Yamilov [14], which are listed in §1 (equations (1.11)-(1.16)). Therefore it may be necessary to seek a change of dependent variables $w = \phi(Q)$ and write the resulting equation in non-potential form. An alternative approach is to apply the Painlevé tests directly

on the semilinear equation, provided that the nonlinear evolution equation is in rational form (i.e., H in (2.43) is a rational function of its arguments).

There are two remarks we wish to make about the above procedure.

1. It is important to first put equation (2.41) into canonical form by making the transformation $v_x = g^{-1/3}(u)$ before applying the pure hodograph transformation (otherwise the partial differential equation will remain quasilinear). To demonstrate this, consider the Harry-Dym equation

$$u_t = (u^{1/2})_{xxx}. \tag{2.44}$$

First put (2.44) into potential form by letting $v_x = u$, then

$$v_t = (v_x^{-1/2})_{xx}. \tag{2.45}$$

Applying a pure hodograph transformation to (2.45) gives

$$r_\tau = (r_\xi^{-1/2})_{\xi\xi},$$

which is just the same equation (i.e., the potential Harry-Dym equation is invariant under a pure hodograph transformation).

2. If the quasilinear partial differential equation is not in the special

form (2.41) then the transformation $v_x = g^{-1/3}(u)$ yields either a higher order or nonlocal partial differential equation. For example, consider the partial differential equation

$$u_t = u^{-3}u_{xxx}. \tag{2.46}$$

Then after making the transformation $v_x = u$ we obtain

$$v_{xt} = v_x^{-3}v_{xxxx},$$

or

$$v_t = v_x^{-3}v_{xxx} + 3\int^x v_x^{-4}v_{xx}v_{xxx}.$$

By considering several examples, we shall now demonstrate how the procedure developed above can be applied to determining whether a given third order quasilinear partial differential equation might be linearizable. In these examples, we apply the Painlevé tests to the semilinear equation to determine necessary conditions for the equation to be possibly linearizable. Furthermore, we show that when these conditions are satisfied, then the equation is equivalent to a linearizable equation by exhibiting the requisite transformation. Since we are using the Painlevé tests in these examples to exclude several possibilities, when we conclude below that an equation is "nonlinearizable" (because the above conditions are not satisfied), we mean "nonlinearizable, subject to the validity of the Painlevé tests", i.e., in these cases the equation is "probably nonlinearizable."

Example 2.1

In this example we determine for which values of the constant α is the equation

$$u_t = u^3 u_{xxx} + \alpha u^2 u_x u_{xx}, \tag{2.47}$$

linearizable. Equation (2.47) was considered by Kawamoto [9], where we note that if $\alpha = 0$, then (2.47) is equivalent to the Harry-Dym equation $v_t + 2(v^{-1/2})_{xxx} = 0$ (set $u = v^{-1/2}$). In order to set (2.47) in canonical form we make the transformation $v_x = 1/u$, hence

$$v_t = v_x^{-3} v_{xxx} - \frac{1}{2}(\alpha + 3) v_x^{-4} v_{xx}{}^2. \tag{2.48}$$

Applying a pure hodograph transformation to (2.48) gives

$$\eta_\tau = \eta_{\xi\xi\xi} + \frac{1}{2}(\alpha - 3)\eta_{\xi\xi}^2 \eta_\xi^{-1} \tag{2.49}$$

We now apply a sequence of transformations to (2.49). First we put (2.49) into non-potential form by letting $w = \eta_\xi$, hence

$$w_\tau = w_{\xi\xi\xi} + \frac{1}{2}(\alpha - 3)(w_\xi^2/w)_\xi . \tag{2.50}$$

Then, in order to determine whether (2.50) is equivalent to one of the six linearizable equations given by Svinolupov, Sokolov and Yamilov [14] (equations (1.11)-(1.16)), we let $Q = \ln w$, hence

$$Q_\tau = Q_{\xi\xi\xi} + \alpha Q_\xi Q_{\xi\xi} + \frac{1}{2}(\alpha - 1)\, Q_\xi^3. \tag{2.51}$$

Finally, putting (2.51) into non-potential form

$$q_\tau = q_{\xi\xi\xi} + \alpha\,(q q_{\xi\xi} + q_\xi^2) + \frac{3}{2}(\alpha - 1) q^2 q_\xi. \tag{2.52}$$

(additionally it is simpler to apply Painlevé analysis on equation (2.52) rather than on (2.50)). It is shown in Appendix B that equation (2.52) can pass the Painlevé tests only if either $\alpha = 0$, $\alpha = 3/2$ or $\alpha = 3$. If $\alpha = 0$, then (2.52) is the MKdV equation, which is known to be linearizable [22]. If $\alpha = 3/2$ or $\alpha = 3$ (after rescaling q), then (2.52) is the second equation in the Burgers' hierarchy

$$q_\tau = q_{\xi\xi\xi} + \frac{3}{2}(q q_{\xi\xi} + q_\xi^2) + \frac{3}{4}\, q^2 q_\xi \tag{2.53}$$

(Olver [25]), which is reduced by the Cole-Hopf transformation

$$q_\tau = 2(\ln u)_\xi = 2u_\xi / u,$$

to the linear partial differential equation

$$u_\tau = u_{\xi\xi\xi}$$

(i.e., equation (2.53) is equivalent to (1.11)). Therefore we conclude that equation (2.47) is linearizable only for these three values of α.

Example 2.2

Consider the equation

$$u_t = [u_x(1 + u^2)^{-3/2}]_{xx} + 2\alpha u_x(1 + u^2)^{-3/2}, \tag{2.54}$$

where α is a constant. Note that if $\alpha = 0$, then (2.54) is an equation which was shown to be linearizable by Wadati, Konno and Ichikawa [6a]. To put (2.54) into canonical form we make the transformation $v_x = (1 + u^2)^{1/2}$, hence we obtain

$$v_t = v_x^{-3}v_{xxx} - \frac{3}{2}v_x^{-4}v_{xx}^2[(1 - 2v_x^2)/(1 - v_x^2)] - \alpha v_x^{-2}. \tag{2.55}$$

Applying a pure hodograph transformation to (2.55) gives

$$\eta_\tau = \eta_{\xi\xi\xi} + \alpha\eta_\xi^3 + \frac{3}{2}\frac{\eta_\xi\eta_{\xi\xi}}{1 - \eta_\xi^2}$$

which has the non-potential form ($w = \eta_\xi$)

$$w_\tau = w_{\xi\xi\xi} + 3\alpha w^2 w_\xi + \frac{3}{2}[ww_\xi^2/(1 - w^2)]_\xi. \tag{2.56}$$

Equation (2.56) is equivalent to equation (1.19) (after rescaling the variables), which is known as the 'deformed MKdV' equation [17] or 'modified MKdV' equation [18] and as shown in §1, is equivalent to the CDF equation (1.14) via the transformation $w = \cosh(q/2)$. Hence we obtain

$$q_\tau = q_{\xi\xi\xi} - \frac{1}{8}q_\xi^3 + 3\epsilon\sinh^2(q/2)q_\xi,$$

or

$$q_\tau = q_{\xi\xi\xi} - \frac{1}{8}q_\xi^3 + \frac{3}{4}\epsilon(e^q - 2 + e^{-q})q_\xi. \qquad (2.57)$$

If $\epsilon = 0$ then (2.57) is the potential MKdV equation, while if $\epsilon \neq 0$, then (2.57) is the CDF equation. Therefore equation (2.54) is linearizable for all values of ϵ.

Example 2.3

Consider the equation

$$u_t + 2(u^{-1/2})_{xxx} + f'(u^{1/2})u_x = 0, \qquad (2.58)$$

where f is a rational function and prime denotes differentiation with respect to the argument. The objective is to determine for which choices of f is (2.58) linearizable (note that if $f' \equiv 0$, then (2.58) is the Harry-Dym equation). First we put (2.58) into canonical form by making the transformation $v_x = u^{1/2}$; hence we obtain

$$v_t = v_x^{-3}v_{xxx} - \frac{3}{2}v_x^{-4}v_{xx}^2 - f(v_x). \qquad (2.59)$$

Applying a pure hodograph transformation to (2.59) gives

$$r_\tau = r_{\xi\xi\xi} - \frac{3}{2}r_{\xi\xi}^2 r_\xi^{-1} - r_\xi f(r_\xi^{-1}),$$

which has the non-potential form ($w = r_\xi$)

$$w_\tau = w_{\xi\xi\xi} - \frac{3}{2}(w_\xi^2/w)_\xi - g'(w)w_\xi , \tag{2.60}$$

where $g(w): = w\, f(1/w)$. It can be shown that (2.60) can pass the Painlevé tests if and only if

$$g(w) = \alpha w^3 + \beta w + \gamma w^{-1}, \tag{2.61}$$

hence

$$f(w) = \alpha w^{-2} + \beta + \gamma w^2, \tag{2.62}$$

where α, β and γ are arbitrary constants (see Appendix C for details). Note that equation (2.60) with $g(w)$ as given by (2.61) is just equation (1.18), which is equivalent to the CDF equation (1.14) if either $\alpha \neq 0$ or $\gamma \neq 0$ (let $w = e^{u/2}$); if $\alpha = \gamma = 0$ and $q = w_\xi/w$, then q satisfies the MKdV equation, hence equation (2.60) with $g(w)$ as given by (2.61) is linearizable. Therefore, we conclude that the most general equation of the form (2.58) which is linearizable is

$$u_t + 2(u^{-1/2})_{xxx} + 2\gamma u^{1/2} u_x - 2\alpha u^{-3/2} u_x = 0. \tag{2.63}$$

III. HIGHER ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS.

The method developed for second and third order quasilinear partial differential equations can easily be extended to higher order equations.

Proposition 3.1

The most general quasilinear partial differential equation of the form

$$u_t = g(u)u_{nx} + f(u,u_x,\ldots,u_{(n-1)x}), \quad u_{nx} \equiv \frac{\partial^n u}{\partial x^n}, \quad \frac{dg}{du} \neq 0 \tag{3.1}$$

which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form

$$S_\tau = S_{n\xi} + G(S,S_\xi,\ldots,S_{(n-1)\xi}), \tag{3.2}$$

is given by

$$u_t = g(u)u_{nx} + \left(\frac{g''}{g'} - \frac{n+1}{n}\frac{g'}{g}\right)B(u,u_x,\ldots,u_{(n-2)x})u_x$$
$$+ B_u u_x + \sum_{r=2}^{n-1} B_{u_{(r-1)x}} u_{rx} + \left(\frac{gg''}{g'} - \frac{g'}{n}\right)u_x u_{(n-1)x}, \tag{3.3}$$

where prime denotes derivative with respect to u, and g(u) and $B(u,u_x,\ldots,u_{(n-2)x})$ are arbitrary functions. Furthermore, equation (3.2) is equivalent to the equation

$$v_t = v_x^{-n} v_{nx} + H(v_x,v_{xx},\ldots,v_{(n-1)x}), \tag{3.4}$$

which is transformed via a pure hodograph transformation to

$$r_t = r_{n\xi} + H(r_\xi, r_{\xi\xi},\ldots, r_{(n-1)\xi}). \tag{3.5}$$

Proof

The proof is analogous to those for Propositions 2.1 and 2.2 above and so we shall only sketch an outline. In equation (3.1) we make the transformation

$$\tau = t, \; \xi = F(x,t), \; \eta(\xi,\tau) = u(x,t),$$

and choose F such that

$$g F_x^n = 1, \text{ i.e., } F_x = g^{-1/n}, \tag{3.6}$$

$$F_t = A(u,u_x,\ldots,u_{(n-1)x}), \tag{3.7}$$

where $A(u,u_x,\ldots,u_{(n-1)x})$ is such that the compatibility of (3.6),(3.7), i.e. $F_{xt} = F_{tx}$) implies (3.1). Therefore

$$-\frac{1}{n}g^{-1/n}g'u_{xxx} - \frac{1}{n}g^{-(n+1)/n}g'f(u,u_x,\ldots,u_{(n-1)x})$$
$$= A_u u_x + \sum_{r=2}^{n} A_{u_{(r-1)x}} u_{rx} \; . \tag{3.8}$$

Hence

$$A(u,u_x,\ldots,u_{(n-1)x}) = -\frac{1}{n}g^{-(n+1)/n}g'[gu_{(n-1)x}$$
$$+ B(u,u_x,\ldots,u_{(n-2)x})], \tag{3.9}$$

where $B(u,u_x,\ldots,u_{(n-2)x})$ is an arbitrary function. Therefore, from equation (3.9) we find that

$$f(u,u_x,\ldots,u_{(n-1)x}) = \left(\frac{g''}{g'} - \frac{n+1}{n}\frac{g'}{g}\right)B(u,u_x,\ldots,u_{(n-2)x})u_x$$
$$+ B_u u_x + \sum_{r=2}^{n-1} B_{u_{(r-1)x}} u_{rx} + \left(\frac{gg''}{g'} - \frac{g'}{n}\right)u_x u_{(n-1)x}, \tag{3.10}$$

Hence, it follows that the most general equation of the form (3.10) which is transformed via an extended hodograph transformation into a semilinear partial differential equation has the form (3.3) as required. Equation (3.4) is obtained from (3.3) by making the transformation $v_x = g^{-1/n}(u)$, where $H(v_x,\ldots,v_{(n-1)x})$ is expressible in terms of $B(u,u_x,\ldots,u_{(n-2)x})$ and $g(u)$ and therefore is the canonical equation. Finally, equation (3.5) is obtained by applying a pure hodograph transformation to (3.12).

Proposition 3.1 provides an algorithmic method of transforming the general quasilinear partial differential equation

$$u_t = g(u)u_{nx} + f(u,u_x,\ldots,u_{(n-1)x}) \tag{3.11a}$$

where

$$f(u,u_x,\ldots,u_{(n-1)x}) = \left(\frac{g''}{g'} - \frac{n+1}{n}\frac{g'}{g}\right)B(u,u_x,\ldots,u_{(n-2)x})u_x$$
$$+ B_u u_x + \sum_{r=2}^{n-1} B_{u_{(r-1)x}} u_{rx} + \left(\frac{gg''}{g'} - \frac{g'}{n}\right)u_x u_{(n-1)x}, \tag{3.11b}$$

into a semilinear partial differential equation as follows:

1. Put equation (3.11) into the potential canonical form by making the transformation $v_x = g^{-1/n}(u)$; hence we obtain

$$v_t = v_x^{-n} v_{nx} + H(v_x, v_{xx}, \ldots, v_{(n-1)x}). \tag{3.12}$$

2. Apply a pure hodograph transformation to equation (3.12); hence we obtain

$$\eta_t = \eta_{n\xi} + \tilde{H}(\eta_\xi, \eta_{\xi\xi}, \ldots, \eta_{(n-1)\xi}). \tag{3.13}$$

3. The resulting partial differential equation will be in potential form and usually one first puts the equation into nonpotential form by making the transformation $w = \eta_\xi$. It may also be convenient to seek a change of dependent variables $w = \phi(Q)$ (and then write the resulting equation in non-potential form if necessary) and then apply the Painlevé tests to the semilinear equation to determine if it is possibly linearizable. (For fourth and higher order semilinear partial differential equations, there is, at present, no equivalent theorem to the one given by Svinolupov, Sokolov and Yamilov [14] for third order equations.)

Example 3.1

In this example we consider the equation

$$u_t = u^{5/2}u_{5x}, \tag{3.14}$$

which was shown by Konopelchenko and Dubrovsky [26] to be the compatibility condition of the linear operators

$$L = u^{3/2}\partial_x^3,$$

$$M = 9u^{5/2}\partial_x^5 + \frac{45}{2}u^{3/2}u_x\partial_x^4 + 15u^{3/2}u_{xx}\partial_x^3 + \partial_t,$$

where $\partial_x \equiv \partial/\partial x$, $\partial_t \equiv \partial/\partial t$ (i.e., $LM - ML = 0$ if and only if u satisfies (3.13)).

We first put (3.14) into canonical form by making the transformation $v_x = u^{-1/2}$, hence we obtain

$$v_t = v_x^{-5}v_{5x} - 10v_x^{-6}(v_{2x}v_{4x} + v_{3x}^2) + 60v_x^{-7}v_{2x}^2v_{3x} - 45v_x^{-8}v_{xx}^4.$$

Applying a pure hodograph transformation to the above equation we obtain

$$r_t = r_{5\xi} - 5r_{2\xi}r_{4\xi}r_\xi^{-1} + 5r_{2\xi}^2r_{3\xi}r_\xi^{-2}, \tag{3.15}$$

which has the nonpotential form

$$w_t = w_{5\xi} - 5w^{-1}(w_\xi w_{4\xi} + w_{2\xi}w_{3\xi}) + 10w^{-2}(w_\xi^2w_{3\xi} + w_\xi w_{2\xi}^2)$$

$$- 10w^{-3}w_\xi^3w_{2\xi}. \tag{3.16}$$

We now let $Q = \ln w$, hence

$$Q_\tau = Q_{5\xi} + 5Q_{2\xi}Q_{3\xi} - 5Q_\xi Q_{2\xi}^2 - 5Q_\xi^2 Q_{3\xi} + Q_\xi^5,$$

which has the nonpotential form

$$q_\tau = q_{5\xi} + 5q_\xi q_{3\xi} + 5q_{2\xi}^2 - 5q_\xi^3 - 20qq_\xi q_{2\xi} - 5q^2 q_{3\xi} + 5q^4 q_\xi. \qquad (3.17)$$

Equation (3.17) can be transformed into two linearizable fifth order equations. Fordy and Gibbons [27] show that if q satisfies (3.17) and u and v are defined by the Miura transformations

$$u = -q_\xi - q^2, \quad v = q_\xi - \tfrac{1}{2}q^2, \qquad (3.18)$$

then u and v respectively satisfy the Sawada-Kotera equation [28] (sometimes referred to as the Caudrey-Dodd-Gibbon equation [29])

$$u_\tau = u_{5\xi} + 5uu_{3\xi} + 5u_\xi u_{2\xi} + 5u^2 u_\xi, \qquad (3.19)$$

and the Kaup equation [30] (sometimes referred to as the Kuperschmidt equation, cf. [27])

$$v_\tau = v_{5\xi} + 10vv_{3\xi} + 25v_\xi v_{2\xi} + 20v^2 v_\xi. \qquad (3.20)$$

Both equations (3.19) and (3.20) are known to be linearizable, see [31] and [30] respectively. This shows that equation (3.14) is the quasilinear analogue of equation (3.17), which is linearizable and so (3.14) should not be regarded as a "new" linearizable fifth order equation.

Example 3.2

The second equation in the Harry-Dym hierarchy is given by

$$u_t = u^3[u(uu_{xx} - \tfrac{1}{2}u_x^2)]_{xxx}$$

$$= u^5u_{5x} + 5u^4(u_xu_{4x} + u_{xx}u_{3x}) + \tfrac{5}{2}u^3u_x^2u_{3x} \tag{3.21}$$

(see [2b] or [32]). We first put (3.21) into canonical form by making *the transformation* $v_x = u^{-1}$, *hence we obtain*

$$v_t = v_x^{-5}v_{5x} - \tfrac{5}{2}v_x^{-6}(4v_{2x}v_{4x} - 3v_{3x}^2)$$

$$+ \frac{105}{2}v_x^{-7}v_{2x}^2v_{3x} - \frac{315}{8}v_x^{-8}v_{xx}^4. \tag{3.22}$$

Applying a pure hodograph transformation to (3.22) gives

$$\eta_\tau = \eta_{5\xi} - 5\eta_{2\xi}\eta_{4\xi}\eta_\xi^{-1} - \tfrac{5}{2}\eta_{\xi\xi\xi}^2\eta_\xi^{-1}$$

$$+ \frac{25}{2}\eta_{2\xi}^2\eta_{3\xi}\eta_\xi^{-2} - \frac{45}{8}\eta_{\xi\xi}^4\eta_\xi^{-3} \tag{3.23}$$

which has the nonpotential form

$$w_\tau = w_{5\xi} - 5w^{-1}(w_\xi w_{4\xi} + 2w_{2\xi}w_{3\xi}) + \frac{35}{2}w^{-2}w_\xi^2 w_{3\xi}$$

$$+ \frac{55}{2}w_\xi w_{2\xi}^2 w^{-2} - \frac{95}{2}w^{-3}w_\xi^3 w_{2\xi} + \frac{135}{8}w_\xi^5 w^{-4}. \qquad (3.24)$$

As in Example 3.1 above, we now let $Q = \ln w$, hence

$$Q_\tau = Q_{5\xi} - \frac{5}{2}(Q_\xi Q_{2\xi}^2 + Q_\xi^2 Q_{3\xi}) + \frac{3}{8}Q_\xi^5,$$

which has the nonpotential form

$$q_\tau = q_{5\xi} - \frac{5}{2}q_\xi^3 - 10qq_\xi q_{2\xi} - \frac{5}{2}q^2 q_{3\xi} + \frac{15}{8}q^4 q_\xi. \qquad (3.25)$$

Equation (3.25) is the second equation in the MKdV hierarchy (see[25]).

This provides further evidence of the close relationship between the Harry-Dym equation and the MKdV equation. It is well known that the inverse scattering schemes for the MKdV equation and the Harry-Dym equation are related through a sequence of gauge transformations which also involve an interchange of independent and dependent variables [34] (see also [35]). Since the recursion operator for the Harry-Dym equation is well known (cf. [2b], [32], then it can be shown (Fokas and Fuchssteiner [36]) that these recursion operators (or hereditary symmetries in the terminology of [36]) are related by a Bäcklund transformation.

IV. DISCUSSION

In this paper we have discussed the relationship between quasilinear and semilinear partial differential equations. In particular, an algorithmic procedure was developed for finding the quasilinear (semilinear) analogue of a given semilinear (quasilinear) equation (if it exists). Furthermore, the associated quasilinear (semilinear) equation is unique up to equivalence. This procedure provides a simple algorithmic method for determining whether a given quasilinear partial differential equation might be linearizable. Consequently, several quasilinear partial differential equations which might appear initially to be "new" linearizable equations are actually equivalent to the quasilinear analogue of a semilinear equation which is known to be integrable.

For example, Abellanas and Galindo [37] showed that the quasilinear equation

$$u_t = (\alpha u^2 + 2\beta u + \gamma)^{3/2} u_{xxx}, \tag{4.1}$$

where α, β, γ are constants, possesses a bihamiltonian structure and hence an infinite number of nontrivial conservation laws. Note that equation (4.1) contains as special cases both the Harry-Dym equation

$$u_t = u^3 u_{xxx}, \tag{4.2}$$

and an equation considered by Bruschi and Ragnisco [38]

$$u_t = u^{3/2} u_{xxx}, \tag{4.3}$$

Applying the method developed in the present paper shows that (4.1) is transformed into either the MKdV equation (if $\alpha \neq 0$) or the linear equation $\eta_\tau = \eta_{\xi\xi\xi}$ (if $\alpha = 0$ and $\beta \neq 0$). (Bruschi and Ragnisco [38] showed that (4.3) can be transformed via an extended hodograph transformation to the linear equation.)

In two recent papers, Mikhailov and Shabat [39] have determined necessary conditions for the existence of nontrivial conservation laws for systems of equations of the form

$$\underline{u}_t = A(\underline{u})\underline{u}_{xx} + \underline{f}(\underline{u},\underline{u}_x), \tag{4.4}$$

where

$$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A(\underline{u}) = \begin{pmatrix} a(u,v) & b(u,v) \\ c(u,v) & d(u,v) \end{pmatrix},$$

$$\underline{f}(\underline{u},\underline{u}_x) = \begin{pmatrix} f(u,v,u_x,v_x) \\ g(u,v,u_x,v_x) \end{pmatrix}.$$

(This is analogous to the work of Svinolupov, Sokolov and Yamilov [14] who also used the existence of nontrivial conservation laws as the criterion in their determination of which third order semilinear equations are linearizable.) In order to determine their necessary conditions, Mikhailov and Shabat [39] first transformed the quasilinear equation (4.4) into the semilinear canonical form

$$\underline{\eta}_\tau = \sigma_3 \underline{\eta}_{\xi\xi} + H(\underline{\eta}, \underline{\eta}_\xi), \tag{4.5}$$

where

$$\underline{\eta} = \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$H(\underline{\eta}, \underline{\eta}_\xi) = \begin{pmatrix} h(\eta, \psi, \eta_\xi, \psi_\xi) \\ k(\eta, \psi, \eta_\xi, \psi_\xi) \end{pmatrix}. \tag{4.6}$$

This transformation was achieved by first transforming (4.4) into the form

$$\underline{U}_t = g(\underline{U})\sigma_3 \underline{U}_{xx} + \underline{F}(\underline{U}, \underline{U}_x), \tag{4.7}$$

where

$$\underline{U} = \begin{pmatrix} U \\ V \end{pmatrix}, \quad \underline{F}(\underline{U}, \underline{U}_x) = \begin{pmatrix} F(U, V, U_x, V_x) \\ G(U, V, U_x, V_x) \end{pmatrix}$$

(so equations (4.4) and (4.7) are equivalent), and then applying an extended hodograph transformation to (4.7).

We note that it would be useful to extend the method outlined in earlier sections to quasilinear nonlinear evolution equations in two spatial and one temporal dimensions. Due to the presence of more independent variables, there is more flexibility in the hodograph transformation.

Finally, we make a remark regarding the application of the Painlevé tests. These tests have proved to be a useful criterion for the indentification of linearizable (semilinear) partial differential equations; however, there is one major restriction in their application. Since the Painlevé tests require that a linearizable partial differential equation possesses the Painlevé property possibly after a change of variables, then one may first have to make a change of variables before applying the tests. An open question is: Which transformations are allowable in the application of the Painlevé tests? (i.e., which transformations does one have to check?). We believe that pure hodograph transformations and the notion of equivalence are useful tools in this direction.

APPENDIX A

In this appendix we show that the partial differential equation

$$u_t = u_{xx} + h(u)u_x, \tag{A.1}$$

where $h(u)$ is a rational function of u can pass the Painlevé tests if and only if $h(u)$ is a linear function of u. In (A.1) consider the traveling wave solution $u(x,t) = u(z)$, $z = x-ct$, where c is a constant. Then $u(z)$ satisfies

$$u'' + h(u)u' + cu' = 0. \tag{A.2}$$

Integrating yields

$$u' + H(u) + cu = A, \tag{A.3}$$

where $\frac{dH}{du} = h(u)$ and A is a constant. It is known that the only equation of the form

$$u' = R(u),$$

where $R(u)$ is a rational function of u, which is of Painlevé type is the Riccati equation

$$u' = c_2u^2 + c_1u + c_0,$$

where α_2, α_1 and α_0 are constants (see Hille [40] or Ince [41] for a proof). Therefore (A.3) is of Painlevé type if and only if H(u) is a quadratic function of u, so necessarily

$$h(u) = \alpha u + \beta, \tag{A.4}$$

where α and β are constants. If h(u) has the special form (A.4), then equation (A.1) is either (i) equivalent to Burgers' equation if $\alpha \neq 0$, or (ii) a linear equation if $\alpha = 0$. Hence (A.1) can pass the Painlevé tests if and only if h(u) is a linear function of u, as required.

APPENDIX B

In this appendix we show that the partial differential equation

$$q_t = q_{xxx} + (qq_{xx} + q_x^2) + \frac{3}{2}(\alpha - 1)q^2q_x, \tag{B.1}$$

where α is a constant, can pass the Painlevé tests if and only if α takes one of the three values 0, 3/2, 3. We first note that if $\alpha = 0$ then (B.1) is the MKdV equation, which is known to be linearizable [15] and pass the Painlevé PDE test [22]. Now we shall assume that $\alpha \neq 0$ and we consider the time-independent solution $q(x,t) = y(x)$ of (B.1), then y(x) satisfies

$$y''' + \alpha[yy'' + (y')^2] + \frac{3}{2}(\alpha - 1)y^2y' = 0. \tag{B.2}$$

which can be integrated once, yielding

$$y'' + \alpha yy' + \frac{1}{2}(\alpha - 1)y^3 = A, \tag{B.3}$$

where A is an arbitrary constant. Now make the transformation $y = 3w/\alpha$, giving

$$w'' + 3ww' + \frac{9}{2}(\alpha - 1)\alpha^{-2}w^3 = B, \tag{B.4}$$

where $B := \alpha A/3$. Ince [43, p332] shows that the equation

$$w'' + 3ww' + \gamma w^3 = B, \tag{B.5}$$

where γ and $B(\neq 0)$ are constants, is of Painlevé type if and only if $\gamma = 1$ (the case $B = 0$ is discussed below). Hence (B.4) (and hence also (B.3)) is of Painlevé type if and only if

$$\frac{9}{2}(\alpha - 1) = \alpha^2,$$

i.e.,

$$(\alpha - 3)(\alpha - \frac{3}{2}) = 0. \tag{B.6}$$

If $\alpha = 3/2$ or $\alpha = 3$ (after rescaling q by a factor of 2) then (B.1) is the second equation in the Burger's hierarchy

$$q_t = q_{xxx} = \frac{3}{2}(qq_{xx} + q_x^2) + \frac{3}{4}q^2q_x, \tag{B.7}$$

(Olver [25]), which is reduced by the Cole-Hopf transformation

$$q = 2(\ln u)_x = 2u_x/u,$$

to the linear partial differential equation

$$u_t = u_{xxx}.$$

If $B = 0$ in (B.5), then there exist two choices of γ such that the equation is of Painlevé type, $\gamma = 1$ or $\gamma = -9$. If $\gamma = -9$, then

$$\frac{9}{2}(\alpha - 1) = -9\alpha^2,$$

i.e.,

$$(\alpha + 1)(\alpha - \frac{1}{2}) = 0. \tag{B.8}$$

If $\alpha = -1$ or $\alpha = 1/2$ (after rescaling q by a factor of 1/2), then (B.1) is

$$q_t = q_{xxx} - (qq_{xx} + q_x^2) - 3q^2 q_x. \tag{B.9}$$

If we seek a solution of (B.9) in the form

$$q(x,t) = \phi^p \sum_{j=0}^{\infty} q_j(t)\phi^j(x,t), \tag{B.10}$$

with $\phi = x + f(t)$, in the neighborhood of the noncharacteristic singularity manifold defined by $\phi = 0$, then leading order analysis shows that

$p = -1$ and there are two choices for q_0, $q_0 = -1$ and $q_0 = 2$. Equating coefficients of powers of ϕ determines the recursion relations defining $q_j(t)$, for $j \geq 1$. For to the choice $q_0 = -1$, the resonances are -1, 3, 3 (the resonances are the values of j at which arbitrary functions arise in the expansion (B.10) and for each positive resonance there is a compatibility condition which must be identically satisfied). A double resonance indicates that the expansion (B.10) does not represent the general solution (logarithmic terms must be introduced into the expansion (B.10) so that it represents the general solution). For the choice $q_0 = 2$, the resonances are -1, 3, 6; the compatibility condition corresponding to the resonance $j = 6$ is not identically satisfied which indicates that logarithmic terms again must be introduced into the expansion (B.10). Therefore (B.9) does not pass the Painlevé PDE test.

We therefore conclude that equation (B.1) can pass the Painlevé tests if and only if α takes one of the three values 0, 3/2, 3, as required.

APPENDIX C

In this appendix we show that the partial differential equation

$$w_t = w_{xxx} - \frac{3}{2}(w_x^2/w)_x + g(w)w_x, \tag{C.1}$$

where $g(w)$ is a rational function, can pass the Painlevé tests if and only if

$$g(w) = \alpha w^3 + \beta w + \gamma w^{-1}, \tag{C.2}$$

where α, β and γ are constants. First, consider the time-independent solution $w(x,t) = y(x)$, then y satisfies

$$y''' = \frac{3}{2}[(y')^2/y]' - g(y)y', \tag{C.3}$$

where $' := d/dx$. Integrating (C.3) gives

$$y'' = \frac{3}{2}(y')^2/y - G(y) + A, \tag{C.4}$$

where $\frac{dG}{dy} = g(y)$ and A is a constant. Multiplying $y^{-3}y'$ and integrating again yields

$$\frac{1}{2}y^{-3}(y')^2 = -\int^y v^{-3}G(v)dv - \frac{A}{2}y^{-2} + B, \tag{C.5}$$

where B is another constant. It is well known that the equation

$$(y')^2 = R(y), \tag{C.6}$$

where $R(y)$ is a rational function, is of Painlevé type if and only if $R(y)$ is a polynomial of degree not exceeding 4 (see Hille [40] or Ince [41] for a proof). Hence equation (C.5) is of Painlevé type if and only if

$$-\int^y v^{-3}G(v)dv - \frac{A}{2}y^{-2} + B = y^{-3}(\alpha_4 y^4 + \alpha_3 y^3 + \alpha_2 y^2 + \alpha_1 y + \alpha_0), \tag{C.7}$$

where α_4, α_3, α_2, α_1 and α_0 are constants. Solving (C.7) for $g(y)$ yields

$$g(y) = -3\alpha_4 y^2 + \alpha_2 - 3\alpha_0 y^{-2}. \tag{C.8}$$

If $g(y)$ has the special form (C.8), then equation (C.1) is equation (1.18) which is equivalent to the CDF equation and which is known to pass the Painlevé PDE test [42]. Hence we have the required result.

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